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Collocation schemes for periodic solutions of neutral delay differential equations

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We introduce two collocation schemes for the computation of periodic solutions of neutral delay differential equations (NDDEs): one based on a direct discretisation of the underlying NDDE, and one based on a discretisation of a related delay differential difference equation (i.e., a delay differential equation coupled with a difference equation). Numerical examples are used to demonstrate these schemes and their respective orders of convergence. Both collocation schemes are implemented in DDE-BIFTOOL, a numerical continuation tool for delay equations. Their use in a continuation setting is shown with one- and two-parameter bifurcation studies of a transmission line model.

1 Introduction

Delay differential equations (DDEs) arise in the modelling of many different physical systems, for example, control systems [1], cell biology [2], lasers [3], and population growth [4]. In this paper we consider the general setting of neutral delay differential equations (NDDEs), which may also contain a delayed derivative term. NDDEs are also used in a wide range of applications, particularly the area of transmission line modelling [5–9]. Like DDEs, NDDEs have an infinite dimensional state space but, unlike DDEs, they possess a continuous (essential) spectrum as well as a point spectrum and also their solutions may be non-smooth [10–12].

Software for the simulation of systems incorporating delay is now widely available and quite robust, see for example [13, 14]. Additionally, the recent development of DDE-BIFTOOL [15] and PDDE-CONT [16], numerical continuation packages for DDEs, has enabled in-depth studies of the solutions and bifurcations arising in such systems. At the moment, however, there is a lack of bifurcation analysis tools for NDDEs. The aim of this paper is to extend the functionality of DDE-BIFTOOL to allow one to continue periodic solutions of NDDEs and their bifurcations.

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We consider the periodic solutions of the following explicit NDDE with a single fixed delay

$$\dot{x}(t) = f(x(t), x(t-\tau), \dot{x}(t-\tau)), \qquad x \in \mathbb{R}^n. \tag{1}$$

Generalisation to multiple fixed delays is straight-forward. As mentioned above, the NDDE (1) supports non-smooth solutions. To prevent this occurring we consider only the case where (1) has a strongly stable difference operator [10, 17]; in the simplest case where the delayed derivative term has a constant coefficient this means that the absolute value of this coefficient must be less than one. Most NDDEs arising from applications have this property, hence this places few limitations on the usefulness of the tools presented. To compute periodic solutions of (1) we propose two new collocation schemes. We use collocation because it has proved to be a reliable and efficient method for computing periodic solutions of differential equations. The well known continuation software AUTO [18] uses collocation as standard, as does CONTENT [19], COLSYS (and its variants) [20], DDE-BIFTOOL [15], and PDDE-CONT [16]. The original work on collocation for DDEs is found in [21, 22]. There has been some progress towards the continuation of periodic solutions of NDDEs by different methods, namely linear finite difference approximations [23] and shooting [24]. Neither of these codes are not publicly available.

The two collocation schemes used here to approximate periodic solutions of the NDDE (1) are

(i) *Direct neutral collocation (DNC)*, where we construct collocation equations for (1) directly.

(ii) *Extended neutral collocation (ENC)*. By extending (1) with $y(t) = \dot{x}(t)$ we arrive at the following ODE coupled with a difference equation:

$$\begin{aligned} \dot{x}(t) &= y(t), \\ y(t) &= f(x(t), x(t-\tau), y(t-\tau)). \end{aligned} \tag{2}$$

In section 2 we begin by introducing the notation used in this paper and give a brief background of collocation. The two collocation schemes, direct and extended neutral collocation, are then formally presented in section 3. There we surmise that the extended neutral collocation scheme (ENC) will be slower and less memory efficient than the direct neutral collocation scheme (DNC) due to the additional equations introduced by the ENC scheme. On the other hand, the ENC method is more accurate as the delayed derivative $\dot{x}(t-\tau)$ is available directly as $y(t-\tau)$, thus there is no need to numerically approximate it from x. This is borne out in the numerical experiments performed in section 4. Finally, a numerical bifurcation study of a transmission line model [7] is performed in section 5.

2 Background and notation

In this section we introduce the notation used throughout the paper and outline the basic principles of collocation that are common to both of the schemes presented here. There are four main components of a collocation scheme for DDEs/NDDEs: a mesh, a collocation solution, a set of collocation points, and an interpolation scheme to determine the value of the delayed variable. We now define these terms and the relationships between them.

The basic premise of any collocation scheme is to use a piecewise polynomial, the collocation solution, to approximate the true solution of a differential equation by ensuring that the piecewise polynomial satisfies the differential equation at a discrete set of *collocation points*. Substituting the collocation solution into the underlying differential equation results in a system of algebraic equations, which can then be solved using standard root finding routines (e.g., Newton iteration). The idea is that, as the number of collocation points (and correspondingly the number of algebraic equations) goes to infinity, the collocation solution converges to the true solution. For many linear differential equations convergence of various collocation schemes for periodic solutions has been proved. See for example [22] for the DDE case, but note that due to the inclusion of delayed derivative terms the NDDE case is currently an open problem.

Throughout this paper we follow the notation used in [25]. For a periodic solution we define the *mesh* Π as a set of $L+1$ mesh points over the range $[0,T]$, where T is the period of the solution, given by

$$\Pi := [0 = t_0, t_1, \ldots, t_{L-1}, t_L = T]. \tag{3}$$

For a uniform mesh (i.e., $t_{i+1} - t_i$ is constant) we define the *mesh size* $h = t_{i+1} - t_i$. Further, we denote by π_m the set of vector-valued polynomials with degree of at most m. The *collocation solution* $x(t)$ is a (vector-valued) piecewise polynomial, defined on the mesh Π, of the form

$$x \in C([0,T], \mathbb{R}^n), \qquad x|_{[t_i,t_{i+1}]} \in \pi_m \quad \text{for} \quad i = 0, \ldots, L-1. \tag{4}$$

To fix a particular solution we require the value of x at $n(mL+1)$ appropriately chosen points in time. Thus, we represent the collocation solution x by a set of values $x_{i+\frac{j}{m}}$ at the *representation points* $t_{i+\frac{j}{m}} := t_i + \frac{j}{m}(t_{i+1} - t_i)$, i.e.,

$$x(t) = \sum_{j=0}^{m} x_{i+\frac{j}{m}} P_{i,j}(t) \quad \text{for} \quad t \in [t_i, t_{i+1}] \tag{5}$$

where $P_{i,j}$ is the Lagrange interpolating polynomial given by

$$P_{i,j}(t) = \prod_{r=0, r\neq j}^{m} \frac{t - t_{i+\frac{r}{m}}}{t_{i+\frac{j}{m}} - t_{i+\frac{r}{m}}}. \tag{6}$$

Note that the collocation solution x is continuous for $t \in [0, T]$ but not necessarily continuously differentiable at the mesh points t_i.

If we choose m collocation points per mesh interval $[t_i, t_{i+1}]$ then there are nmL equations to fix $n(mL + 1)$ unknowns and the remaining n equations are provided by the periodic boundary condition $x(T) = x(0)$. The collocation points $c_{i,l}$ on the mesh interval $[t_i, t_{i+1}]$ are given by

$$c_{i,l} = t_i + c_l(t_{i+1} - t_i) \tag{7}$$

where the c_l are fixed *collocation parameters* with $0 \leq c_1 < \cdots < c_m \leq 1$. They are typically chosen to be the roots of the degree m Gauss-Legendre polynomial transformed to $[0, 1]$. For ODEs and DDEs this choice of collocation parameters allows the error estimate (i.e., the difference between the collocation solution and the true solution) to be sharpened [22], typically from $O(h^m)$ to $O(h^{m+1})$. Additionally, for ODEs (and DDEs on constrained meshes, i.e., the time delay tau is an integer multiple of the mesh size h) the use of Gauss-Legendre collocation points gives rise to super-convergence at the mesh points [26, 27] where the error is typically $O(h^{2m})$.

In constrast, when τ is not an integer multiple of h an interpolation scheme is required to approximate the delayed variable (and its derivative) in order to solve the (N)DDE and this interpolation step destroys the super-convergence property. Throughout this paper we use the interpolation scheme

$$x(t - \tau) = \sum_{j=0}^{m} x_{k+\frac{j}{m}} P_{k,j}(t - \tau) \tag{8}$$

where k is an integer such that $t_k \leq t - \tau < t_{k+1}$. Should $t - \tau < 0$ then a periodic extension of Π and x is used. Where the delayed derivative is required, the derivative of the Lagrange interpolating polynomial is used. Several different interpolation schemes have been proposed for non-neutral DDEs in an attempt to improve the error estimates obtained and to retain the property of super-convergence [21]. Examples of these interpolation schemes include using a higher-order interpolation polynomial (using additional mesh points) or interpolating using the nearby collocation points rather than the mesh points [28, 29].

3 NDDE collocation schemes

In this section we describe in detail the two collocation schemes that we have developed for computing periodic solutions of NDDEs. For both, we reformulate the problem as a periodic boundary value problem (BVP) and then derive the collocation equations. We conclude with a brief comparison of the two resulting linearised systems used to solve the BVP.

3.1 *DNC scheme*

Periodic solutions of (1) can be found as solutions of the periodic BVP

$$\dot{x}(t) = Tf\left(x(t), x((t - T^{-1}\tau) \bmod 1), T^{-1}\dot{x}((t - T^{-1}\tau) \bmod 1)\right) \tag{9a}$$

$$x(0) = x(1), \tag{9b}$$

$$0 = p(x, t), \tag{9c}$$

where T is the period of the solution and p is a suitable phase condition. Time has been rescaled so that the problem is posed on $t \in [0, 1]$; this allows the period of the solution to be determined as an additional unknown parameter. The periodicity condition (9b) is an n-dimensional algebraic condition and so does not ensure that the solutions are truly periodic; this would require the condition $x(t) = x(1 + t)$ for $t \in [-T^{-1}\tau, 0]$. Instead, we use the modulo operator so that values of $t - T^{-1}\tau < 0$ are mapped back onto $[0, 1]$, thus enforcing periodicity.

To derive the collocation equations for (9) we substitute in values for x and its derivatives from (5) and (8). Evaluating the resulting algebraic equation at the m collocation points in each of the L mesh intervals gives the nmL collocation equations

$$\begin{aligned}&\textstyle\sum_{j=0}^{m} P'_{i,j}(c_{i,l})x_{i+\frac{j}{m}} = \\ &Tf\left(\textstyle\sum_{j=0}^{m} P_{i,j}(c_{i,l})x_{i+\frac{j}{m}}, \sum_{j=0}^{m} P_{k,j}(\tilde{c}_{i,l})x_{k+\frac{j}{m}}, T^{-1}\sum_{j=0}^{m} P'_{k,j}(\tilde{c}_{i,l})x_{k+\frac{j}{m}}\right), \\ &\qquad\qquad i = 0, \dots, L-1, \quad l = 1, \dots, m,\end{aligned} \tag{10}$$

where $c_{i,l}$ is as defined in (7), $\tilde{c}_{i,l} = (c_{i,l} - T^{-1}\tau) \bmod 1$ and k is an integer such that $t_k \leq \tilde{c}_{i,l} < t_{k+1}$. To apply Newton iteration to (10), we write it in the form $g_u(u)\Delta u = g(u)$ where $u = \{x_{i+\frac{j}{m}}, T\}$. The resulting linear system is

$$\begin{aligned}
&\textstyle\sum_{j=0}^{m} P'_{i,j}(c_{i,l})\Delta x_{i+\frac{j}{m}} - f(\alpha)\Delta T - TA_0(\alpha)\sum_{j=0}^{m} P_{i,j}(c_{i,l})\Delta x_{i+\frac{j}{m}} \\
&\quad - TA_1(\alpha)\left(\textstyle\sum_{j=0}^{m} P_{k,j}(\tilde{c}_{i,l})\Delta x_{k+\frac{j}{m}} + T^{-2}\tau\left(\sum_{j=0}^{m} P'_{k,j}(\tilde{c}_{i,l})x_{k+\frac{j}{m}}\right)\Delta T\right) \\
&\qquad - TA_2(\alpha)\left(\textstyle\sum_{j=0}^{m} P'_{k,j}(\tilde{c}_{i,l})\Delta x_{k+\frac{j}{m}}\right. \\
&\quad \left.-T^{-2}\left(\textstyle\sum_{j=0}^{m} P'_{k,j}(\tilde{c}_{i,l})x_{k+\frac{j}{m}} + T^{-1}\tau\left(\sum_{j=0}^{m} P''_{k,j}(\tilde{c}_{i,l})x_{k+\frac{j}{m}}\right)\right)\Delta T\right) \\
&\qquad\qquad = -\textstyle\sum_{j=0}^{m} P'_{i,j}(c_{i,j})x_{i+\frac{j}{m}} + Tf(\alpha) \qquad (11)
\end{aligned}$$

where α is a collection of collocation polynomials, which for the DNC scheme takes the form

$$\alpha = \left(\textstyle\sum_{j=0}^{m} P_{i,j}(c_{i,l})x_{i+\frac{j}{m}}, \sum_{j=0}^{m} P_{k,j}(\tilde{c}_{i,l})x_{k+\frac{j}{m}}, T^{-1}\sum_{j=0}^{m} P'_{k,j}(\tilde{c}_{i,l})x_{k+\frac{j}{m}})\right) \qquad (12)$$

and

$$A_0(\xi,\eta,\gamma) = \frac{\partial}{\partial\xi} f(\xi,\eta,\gamma), \qquad (13a)$$

$$A_1(\xi,\eta,\gamma) = \frac{\partial}{\partial\eta} f(\xi,\eta,\gamma), \qquad (13b)$$

$$A_2(\xi,\eta,\gamma) = \frac{\partial}{\partial\gamma} f(\xi,\eta,\gamma). \qquad (13c)$$

We now have nmL algebraic equations and $n(mL+1)+1$ unknowns. The remaining equations are provided by the n periodicity conditions (9b) and the phase condition (9c). Consequently, for a sufficiently fine mesh (i.e., for L sufficiently large) and an initial guess close to a periodic solution of (1), the Newton iteration will converge to a unique collocation solution that approximates the true periodic solution.

3.2 *ENC scheme*

Alternatively, periodic solutions of (2) can be found as solutions of the following periodic BVP:

$$\dot{x}(t) = Ty(t), \tag{14a}$$

$$0 = y(t) - f\left(x(t), x((t - T^{-1}\tau) \bmod 1), y((t - T^{-1}\tau) \bmod 1)\right) \tag{14b}$$

$$x(0) = x(1), \tag{14c}$$

$$y(0) = y(1), \tag{14d}$$

$$0 = p(x, y, t), \tag{14e}$$

where T is the period of the solution and p is a suitable phase condition. To derive the collocation equations for (14a) and (14b) we follow the same procedure as for the DNC scheme. Substituting (5) and (8) into (14a) and (14b), and then evaluating at the m collocation points in each of the L mesh intervals gives the following collocation equations:

$$\begin{aligned}
\textstyle\sum_{j=0}^{m} P'_{i,j}(c_{i,l})x_{i+\frac{j}{m}} &= T\left(\textstyle\sum_{j=0}^{m} P_{i,j}(c_{i,l})y_{i+\frac{j}{m}}\right), \\
0 = Tf\Big(\textstyle\sum_{j=0}^{m} P_{i,j}(c_{i,l})x_{i+\frac{j}{m}}, &\textstyle\sum_{j=0}^{m} P_{k,j}(\tilde{c}_{i,l})x_{k+\frac{j}{m}}, \sum_{j=0}^{m} P_{k,j}(\tilde{c}_{i,l})y_{k+\frac{j}{m}}\Big), \\
& i = 0, \ldots, L-1, \quad l = 1, \ldots, m.
\end{aligned} \tag{15}$$

The corresponding linear system is given by

$$\begin{aligned}
\textstyle\sum_{j=0}^{m} P'_{i,j}(c_{i,l})\Delta x_{i+\frac{j}{m}} - \sum_{j=0}^{m} P_{i,j}(c_{i,l})\Delta y_{i+\frac{j}{m}} - \left(\sum_{j=0}^{m} P_{i,j}(c_{i,l})y_{i+\frac{j}{m}}\right)\Delta T \\
= -\textstyle\sum_{j=0}^{m} P'_{i,j}(c_{i,l})x_{i+\frac{j}{m}} + \sum_{j=0}^{m} P_{i,j}(c_{i,l})\Delta y_{i+\frac{j}{m}},
\end{aligned} \tag{16a}$$

$$\begin{aligned}
&\textstyle\sum_{j=0}^{m} P_{i,j}(c_{i,l})\Delta y_{i+\frac{j}{m}} - A_0(\alpha)\sum_{j=0}^{m} P_{i,j}(c_{i,l})\Delta x_{i+\frac{j}{m}} \\
&\quad - A_1(\alpha)\left(\textstyle\sum_{j=0}^{m} P_{k,j}(\tilde{c}_{i,l})\Delta x_{k+\frac{j}{m}} + T^{-2}\tau\left(\sum_{j=0}^{m} P'_{k,j}(\tilde{c}_{i,l})x_{k+\frac{j}{m}}\right)\Delta T\right) \\
&\quad - A_2(\alpha)\left(\textstyle\sum_{j=0}^{m} P_{k,j}(\tilde{c}_{i,l})\Delta y_{k+\frac{j}{m}} + T^{-2}\tau\left(\sum_{j=0}^{m} P'_{k,j}(\tilde{c}_{i,l})y_{k+\frac{j}{m}}\right)\Delta T\right) \\
&\qquad = -\textstyle\sum_{j=0}^{m} P_{i,j}(c_{i,j})y_{i+\frac{j}{m}} + f(\alpha),
\end{aligned} \tag{16b}$$

where α is a collection of collocation polynomials, which for the ENC scheme takes the form

$$\alpha = \left(\sum_{j=0}^{m} P_{i,j}(c_{i,l})x_{i+\frac{j}{m}}, \sum_{j=0}^{m} P_{k,j}(\tilde{c}_{i,l})x_{k+\frac{j}{m}}, \sum_{j=0}^{m} P_{k,j}(\tilde{c}_{i,l})y_{k+\frac{j}{m}})\right), \quad (17)$$

and $A_{0,1,2}$ are as defined in (13a). We thus have $2nmL$ algebraic equations and $2n(mL+1)+1$ unknowns. As before, the remaining equations are provided by the $2n$ periodicity conditions (14c) and (14d) and the phase condition (14e). This again means that Newton iteration will converge to a collocation solution approximating the true solution.

3.3 *Comparison of the linear systems*

The linear systems resulting from these two schemes are very similar; both consist of a large block matrix, $nmL \times n(mL + 1)$ for the DNC scheme and $2nmL \times 2n(mL+1)$ for the ENC scheme, bordered by one column and either $n+1$ rows for the DNC scheme or $2n+1$ rows for the ENC scheme. Each matrix contains a diagonal band and a circular off-diagonal band corresponding to the delay term. The diagonal bands themselves are composed of $nm \times n(m + 1)$ blocks for the DNC scheme and $2nm\times 2n(m+1)$ for the ENC scheme. Note that the structure of the linear system produced by the DNC scheme is identical to that produced by the DDE scheme described in [21].

The principal difference between the DNC and the ENC scheme is that the former relies on second derivatives of the interpolating polynomials whereas the latter needs only first derivatives. This is because the ENC scheme has direct access to the derivatives of x via the y variable. Numerical experiments (described in the next section) show that the need for the second derivative results in a decrease in the order of convergence of the DNC scheme as compared to the ENC scheme.

In either scheme, whenever $\tilde{c}_{i,l} = t_k$, i.e., $c_{i,l} - T^{-1}\tau$ is a mesh point, (10) and (15) are non-differentiable in the sense that they have different left and right derivatives with respect to T. In this case, we consistently use the right-hand derivative. This non-differentiability may reduce the order of convergence of the Newton iterations used. However, this is not found to have a significant effect on the test cases used in this paper, possibly due to the fact that the periodic solutions considered have continuous derivatives and so any jumps in the derivatives of the collocation solution are small.

4 Numerical examples

Collocation is typically used in a continuation setting (e.g., AUTO [18] or DDE-BIFTOOL [15]), where the aim is to numerically continue periodic solutions and their bifurcations and so understand the dynamics of the system in question. In such a setting it is commonplace to use the techniques of adaptive meshing and pseudo-arclength parameterisation. Also, to continue bifurcations additional test functions must be satisfied, enlarging the BVP further. However, before the collocation schemes proposed in this paper are used in the general setting of continuation we must first determine how well the collocation solutions approximate the true solutions of an NDDE. To do this we consider the simplest case of approximating a periodic solution on a fixed uniform mesh where we keep all parameters of the system constant. By calculating the error between the collocation solution and the reference solution for different mesh sizes we are then able to calculate the rate of convergence of the two schemes. Throughout we use the following NDDE problems as examples:

(E1) The linear non-autonomous equation

$$\dot{x}(t) + \tfrac{1}{4}\dot{x}(t-1) = x(t) + x(t-1) + \sin(t). \tag{18}$$

This equation has an explicit solution of the form $x(t) = A\sin(t) + B\cos(t)$ [23,30] and so the rate of convergence of the numerical solution to the exact solution can be measured directly.

(E2) The nonlinear harmonic oscillator

$$\ddot{x}(t) + b\dot{x}(t) + ax(t) = -\frac{c}{1 + \exp[4\ddot{x}(t-\tau)]}. \tag{19}$$

This equation has been used in [24, 31] as a test case for other methods of finding periodic solutions of NDDEs. When $b = 0$ it is known to possess non-smooth solutions in some regions of parameter space caused by the essential spectrum of the solution passing through the unit disc. We use the parameters investigated in [24], which are $a = 1$, $b = 0.5$, $c = -1.2$, and $\tau = 7.1$.

(E3) The transmission line model

$$C\dot{x}(t) = -\left(\frac{1}{R_c} - \frac{1}{R}\right)x(t) + \left(\frac{1}{R_c} + \frac{1}{R}\right)g(x(t-\tau)) - Cg'(x(t-\tau))\dot{x}(t-\tau), \tag{20}$$

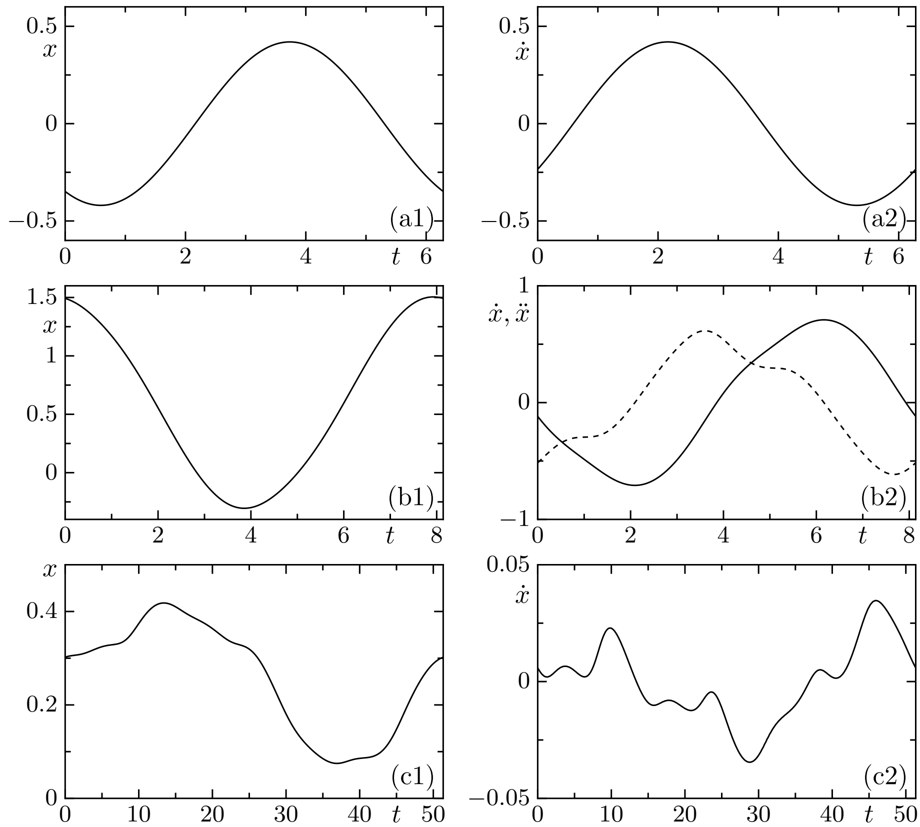

Figure 1. The reference solutions x_{ref} of systems E1, E2, and E3 are shown in the left-hand column and the corresponding derivatives are shown in the right-hand column. Row (a) shows the reference solution of E1 and its derivative. Row (b) shows the reference solution of E2 and its first (solid line) and second (dashed line) derivatives for $a = 1$, $b = 0.5$, $c = -1.2$, and $\tau = 7.1$. Row (c) shows the reference solution of E3 and its derivative for $C = 0.08$, $R_c = 50$, $R = 73.593$, $\tau = 14.6$, $I_s = 5.5 \times 10^{-2}$, and $V_0 = 8 \times 10^{-6}$.

where $y = g(z)$ is the real solution of

$$-\frac{1}{R_c}[y - z] = I_s(\exp((y + z)/V_0) - 1). \tag{21}$$

This system models an electrical device used to produce radio-frequency chaos, developed by Blakely and Corron [7]. Experiments with the physical device indicate a period-doubling route to chaos. Numerical simulations of (20) show good agreement to the experimental data and suggest that the model also possesses a period doubling route to chaos [7]. We use the parameters given in [7], namely $C = 0.08$, $R_c = 50$, $R = 73.593$, $\tau = 14.6$, $I_s = 5.5 \times 10^{-2}$, and $V_0 = 8 \times 10^{-6}$.

Throughout we measure the error of a collocation solution x by using the L_2-norm

$$\|x\| = \left(\int_0^1 (x(t) - x_{\text{ref}}(t))^2 \mathrm{d}t \right)^{\frac{1}{2}}, \tag{22}$$

because of its extensive use in other numerical continuation packages such as AUTO [18]. Note that the L_2-norm used here is defined over the whole solution (a *continuous* norm). Note that it is also quite common to define a norm only at the mesh points of the solution (a *discrete* norm). While this norm show the effects of super-convergence clearly it does not always give a good indication of the "quality" of the overall solution.

Further, note that (22) makes use of a reference solution x_{ref} that is assumed to be as close to the exact solution as possible. For system E1, an exact solution is known and so that is used. However, for systems E2 and E3 exact solutions are not known and instead a collocation solution of very high accuracy, namely with quintic polynomials and 1000 mesh points, is used. The reference solutions used for systems E1, E2, and E3 are shown in the left-hand column of figure 1 and the derivatives of the reference solutions are shown in the right-hand column (the second derivative of x_{ref} is also shown as a dashed line for system E2).

For both DNC, and ENC schemes and each of the systems E1, E2, and E3, we calculate collocation solutions with the number of mesh points varying between $L = 10, \ldots, 500$ (i.e., mesh sizes of $1/500 \leq h \leq 1/10$) and the degree of the polynomials varying between $m = 2, \ldots, 5$. For each value of L and m an initial collocation solution is constructed from the reference solution; this ensures that there is practically no phase shift between the reference solution and the final collocation solution. The initial collocation solution is then corrected using Newton iteration as defined in (11) and (16) for the respective schemes. The collocation solution is deemed to have converged when the Euclidean norm of the right-hand side of (11) or equivalently (16) (the *residual*) is $< 10^{-10}$. The error $\|x\|$ between the collocation solution and the reference solution is then calculated using (22).

In figures 2 and 3 the error $\|x\|$ in the collocation solution is plotted against the mesh size h for the DNC scheme and the ENC scheme respectively. In both figures the different degrees of polynomials used are quadratic ($\triangle$), cubic ($+$), quartic ($\circ$), and quintic ($\times$). Tables 1 and 2 show the orders of convergence of both schemes with respect to the mesh size as determined by a least-squares fit of the data shown in figures 2 and 3. From figures 2 and 3 we see that both schemes for quadratic, cubic, and quartic polynomials converge to the reference solution as the mesh size is decreased. From the numerical data in table 1 the DNC scheme converges at a rate of approximately $O(h^m)$, where m

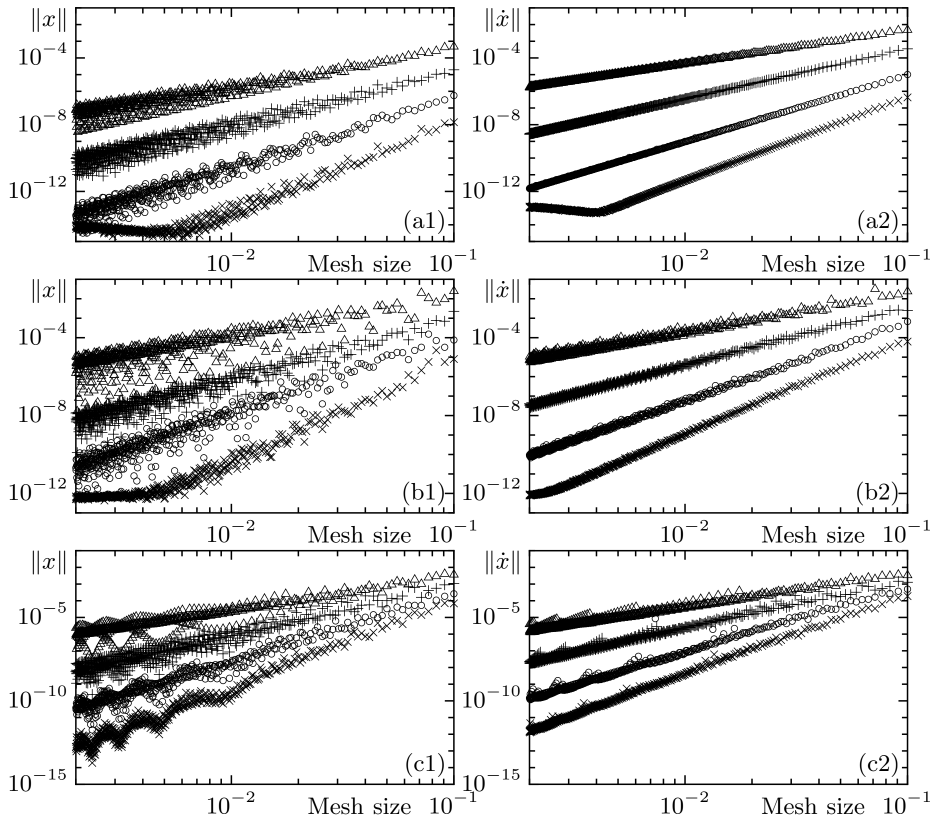

Figure 2. The error of the DNC scheme plotted against the mesh size h for system E1 (row (a)), system E2 (row (b)), and system E3 (row (c)) using quadratic (△), cubic (+), quartic (∘), and quintic (×) polynomials. The left-hand column shows the error of the solution and the right-hand column shows the error of the derivative (cf., figure 3).

Table 1. Numerically computed orders of convergence for the DNC scheme for quadratic, cubic, quartic, and quintic polynomials with the solution (a1) and derivative (a2) of system E1, the solution (b1) and derivative (b2) of system E2, and the solution (c1) and derivative (c2) of system E3. Cf., figure 2.

Degree	(a1)	(a2)	(b1)	(b2)	(c1)	(c2)
2	2.27	2.01	2.07	2.02	2.11	2.02
3	3.16	3.00	3.06	2.98	3.11	2.94
4	4.16	4.00	4.02	4.02	4.10	4.01
5	5.37	5.01	5.06	4.94	5.23	4.90

is the order of the polynomial used. The numerical data in table 2 shows that the ENC scheme converges at a rate of approximately $O(h^{m+1})$, one order higher than the DNC scheme. These estimates on the convergence rates of the two schemes are in line with the previous error estimates of $O(h^{m+1})$ for similar collocation schemes for DDEs [22]; the DNC scheme is one order lower than the ENC scheme due to the approximation of the delayed derivative from

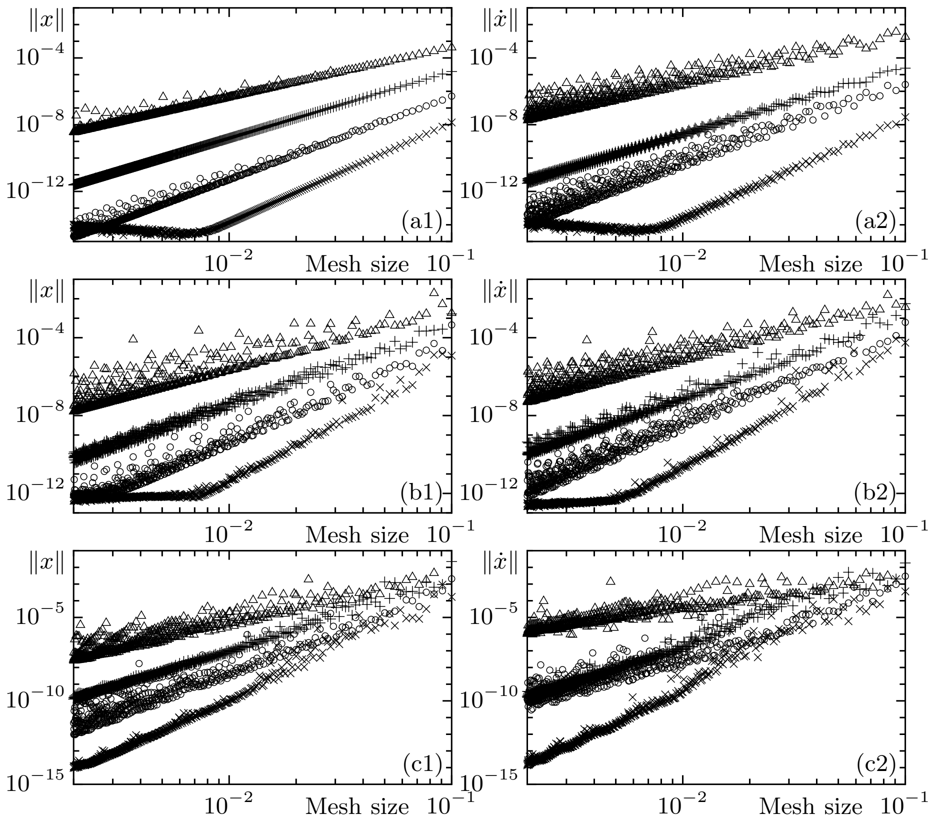

Figure 3. The error of the ENC scheme plotted against the mesh size h for system E1 (row (a)), system E2 (row (b)), and system E3 (row (c)) using quadratic ($\triangle$), cubic ($+$), quartic ($\circ$), and quintic ($\times$) polynomials. The left-hand column shows the error of the solution and the right-hand column shows the error of the derivative (cf., figure 2).

Table 2. Numerically computed orders of convergence for the ENC scheme for quadratic, cubic, quartic, and quintic polynomials with the solution (a1) and derivative (a2) of system E1, the solution (b1) and derivative (b2) of system E2, and the solution (c1) and derivative (c2) of system E3. Cf., figure 3.

Degree	(a1)	(a2)	(b1)	(b2)	(c1)	(c2)
2	2.98	2.90	3.03	2.91	3.12	2.23
3	4.00	4.00	4.02	4.11	4.29	4.64
4	4.90	4.84	4.65	4.79	4.71	3.86
5	6.00	6.00	6.18	6.61	6.37	6.76

the solution profile.

When using quintic polynomials both schemes have a saturation point at which the error in the collocation solution no longer decreases with the decreasing mesh size. In fact, for system E1 the error starts to increase with decreasing mesh size. This is due to the limitations of the double precision arithmetic that was used. When the error in the collocation solution is very small (i.e., $\|x\| < 10^{-13}$) the errors in the linear solve step of the Newton iteration, which grow as the mesh size is increased, become comparatively large. Consequently, when considering system E1, for which the reference solution is the exact solution, there is a mesh size h^* of finite size that gives the smallest error. For $h < h^*$ the error grows due to the errors committed in the linear solve step. For systems E2 and E3, the error saturates as the mesh size is decreased but it does not increase because the reference solution is also a collocation solution and so decreasing the mesh size will eventually result in the reference solution.

Figure 2 shows that there is a large envelope of error values $\|x\|$ for the DNC scheme that increases as the mesh size is decreased. The error is at a minimum when the mesh is constrained, that is, the delay time τ is an integer multiple of the mesh size h (this is the case where no interpolation is necessary to calculate the value of the delayed variable at the collocation points, i.e., the condition necessary for super-convergence). Similarly, the error is at a maximum when the interpolation error is at its largest. As the complexity of the solutions increases, from the solution of system E1 to the solution of system E3, the effects of using a constrained mesh become more apparent.

The effects of using constrained meshes with the ENC scheme are not so clear. An envelope of error values similar to that of the DNC scheme exists when using quadratic and quartic polynomials, however the error values for cubic and quintic polynomials are far more tightly bounded. Additionally, the rates of convergence of the polynomials of even degree are slightly less than expected; this is most noticeable for the solution of system E3 when using quartic polynomials. The reasons for this phenomenon are currently unknown and remain a topic for further research.

5 Bifurcation analysis of the transmission line model E3

The collocation schemes described in this paper have been incorporated into DDE-BIFTOOL, thus allowing numerical bifurcation studies of periodic solutions of NDDEs. In this section we demonstrate the capabilities of these new routines. Specifically, we use numerical continuation to show that the transmission line model E3 does indeed possess a period-doubling route to chaos. The only previous evidence for this is from simulation of system E3 [7]. To continue the solutions of system E3 it is sufficient to use the DNC collocation scheme but we also used the ENC scheme check the results.

Experimental work on the device described by the system E3 indicates that the route to chaos is via a Hopf bifurcation followed by a sequence of period-doubling bifurcations [7]. This device is modelled as a transmission line governed by the Telegrapher's equations (two coupled partial differential equa-

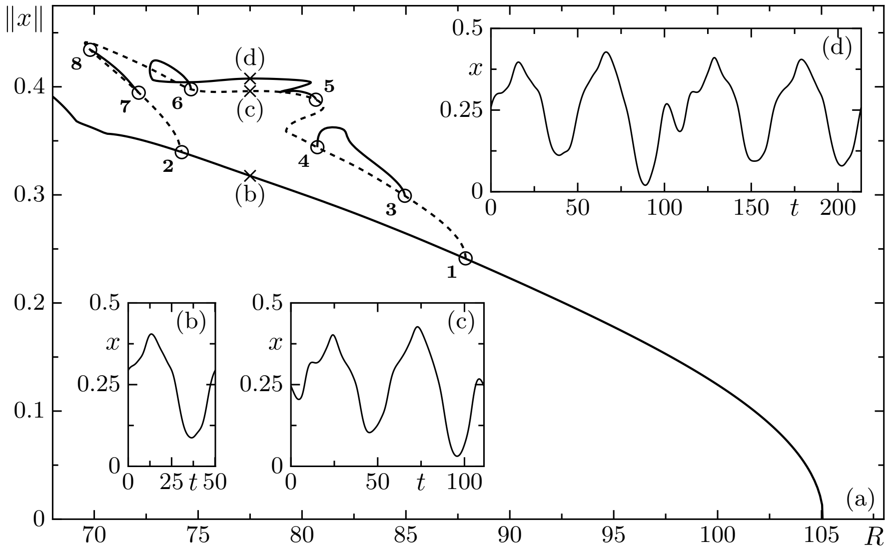

Figure 4. Continuation using the DNC scheme of the periodic solutions bifurcating from a Hopf bifurcation in the transmission line model E3. A sequence of period-doubling bifurcations is found (numbered, see figure 5, and marked by dots) and the bifurcating branches followed. Solutions on each of the branches at the marked points are shown in panels (b), (c), and (d).

tions) with nonlinear components at each end giving the boundary conditions. These partial differential equations are then reduced exactly to an NDDE by using a travelling wave ansatz and applying the boundary conditions. The delay in the system arises from the time taken for the travelling wave to pass from one boundary to the other. Transmission line problems of this form have been considered for many decades [5–9], though the analysis of the resulting NDDEs has been restricted primarily to asymptotics and stability calculations around the steady states, and to simulation.

The parameters of interest in the transmission line model are the time delay τ and the resistance R; these are the parameters that are varied experimentally in [7]. Here we only consider the effects of varying the resistance R and the time delay τ (initially fixed at $\tau = 14.6$). We fix the remaining parameters to match the physical parameter chosen in [7], that is, $C = 0.08$, $R_c = 50$, $I_s = 8 \times 10^{-6}$, and $V_0 = 0.055$.

Figure 4 shows the branch of periodic solutions bifurcating from the Hopf bifurcation at $x \equiv 0$ along with the branches bifurcating at period-doubling bifurcations (marked by dots) as the resistance R is varied. The branch that bifurcates from the Hopf bifurcation (the "period-one" branch) undergoes two period-doubling bifurcations. These two period-doubling bifurcations produce a bubble of "period-two" solutions (marked by a dashed line in figure 4), as is often seen when there is a period-doubling route to chaos. There are six period-doubling bifurcations on the period-two branch; each pair of period-doubling

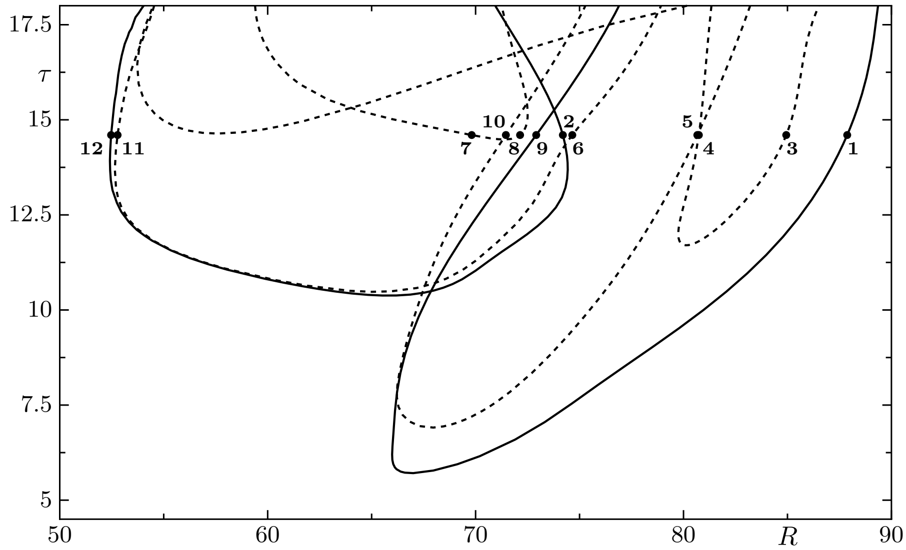

Figure 5. Curves of period-doubling bifurcations of system E3 in the (R, τ)-plane. The numbered points 1–8 correspond to the period-doubling bifurcations in figure 4 for $\tau = 14.6$. None of the apparent branch intersections are codimension 2 bifurcations; all branches are in fact disconnected. The solid curves correspond to the period-doubling bifurcations on the period-one branch and the dashed curves correspond to the period-doubling bifurcations on the period-two branch.

bifurcation produces its own bubble of period-four solutions as shown. Similarly, the branches of period-four solutions are continued and more period-doubling bifurcations are found. The period-eight and higher-period branches are not shown in figure 4 as they are almost indistinguishable from the period-four branches. The process of finding period-doubling bifurcations and following the bifurcating branches can continue until computational limits are reached.

DDE-BIFTOOL is unable to continue bifurcations of periodic solutions in two parameters. Consequently, to achieve this we must append a suitable test function to NDDE E3. The conditions added to the BVP to continue period-doubling bifurcations are

$$\dot{y}(t) = \begin{cases} A_0 y(t) + A_1 y(t-\tau) + A_2 \dot{y}(t-\tau), & t-\tau \geq 0, \\ A_0 y(t) - A_1 y(t-\tau) - A_2 \dot{y}(t-\tau), & t-\tau < 0, \end{cases} \tag{23}$$

where $A_{0,1,2}$ are as defined in (13a). The further conditions are the boundary condition $y(0) = -y(T)$, and a suitable condition for the normalisation of y. Figure 5 shows the branches of period-doubling bifurcations arising from the period-doubling bifurcations shown in figure 4. From figure 5 we see that the period-doubling bifurcations labelled 3 and 4 lie on the same branch as do those labelled 7 and 8. Consequently, as the time delay τ is decreased, the period-doubling bifurcations coalesce. Figure 5 shows that there exists a critical value of τ below which there are no period-doubling bifurcations and, hence, no transition to chaos.

6 Conclusion

In this paper we have presented two collocation schemes for NDDEs: the DNC scheme based on the direct discretisation of the underlying NDDE, and the ENC scheme based on the discretisation of a related delay differential difference equation. We have shown that both schemes perform sufficiently well for use in a numerical continuation setting. Additionally, both schemes have been implemented in DDE-BIFTOOL. Each of the schemes has its advantages and disadvantages; generally speaking the DNC scheme is fast but less accurate, whereas the ENC scheme is slower but more accurate. As such the choice of scheme is problem specific. The reason why the oddness or evenness of the polynomial used effects the convergence of the ENC remains a topic for future research.

The numerical bifurcation study of a transmission line model shows that bifurcation analysis tools for NDDEs can reveal a great deal of useful information about the system in question. There is now the possibility to use these tools to come to a more complete understanding of the dynamics and bifurcations of NDDE models arising in applications.

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